

## Problem 2.10

- (a) Construct  $\psi_2(x)$ .
- (b) Sketch  $\psi_0$ ,  $\psi_1$ , and  $\psi_2$ .
- (c) Check the orthogonality of  $\psi_0$ ,  $\psi_1$ , and  $\psi_2$ , by explicit integration. *Hint:* If you exploit the even-ness and odd-ness of the functions, there is really only one integral left to do.

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## Solution

Here we will solve the Schrödinger equation on the whole line with  $V(x) = (1/2)m\omega^2x^2$ ,

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} + \frac{1}{2}m\omega^2x^2\Psi(x, t), \quad -\infty < x < \infty,$$

with the usual boundary conditions:  $\Psi(x, t)$  and its derivatives tend to zero as  $x \rightarrow \pm\infty$ . Because the Schrödinger equation is linear and homogeneous, the method of separation of variables can be applied. Assume a product solution of the form  $\Psi(x, t) = \psi(x)\phi(t)$  and plug it into the PDE.

$$i\hbar \frac{\partial}{\partial t}[\psi(x)\phi(t)] = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2}[\psi(x)\phi(t)] + \frac{1}{2}m\omega^2x^2[\psi(x)\phi(t)]$$

Evaluate the derivatives.

$$i\hbar\psi(x)\phi'(t) = -\frac{\hbar^2}{2m}\psi''(x)\phi(t) + \frac{1}{2}m\omega^2x^2\psi(x)\phi(t)$$

Divide both sides by  $\psi(x)\phi(t)$  to separate variables.

$$i\hbar \frac{\phi'(t)}{\phi(t)} = -\frac{\hbar^2}{2m} \frac{\psi''(x)}{\psi(x)} + \frac{1}{2}m\omega^2x^2$$

The only way a function of  $t$  can be equal to a function of  $x$  is if both are equal to a constant  $E$ .

$$i\hbar \frac{\phi'(t)}{\phi(t)} = -\frac{\hbar^2}{2m} \frac{\psi''(x)}{\psi(x)} + \frac{1}{2}m\omega^2x^2 = E$$

As a result of applying the method of separation of variables, Schrödinger's equation has reduced to two ODEs—one in each of the independent variables,  $x$  and  $t$ .

$$\left. \begin{aligned} i\hbar \frac{\phi'(t)}{\phi(t)} &= E \\ -\frac{\hbar^2}{2m} \frac{\psi''(x)}{\psi(x)} + \frac{1}{2}m\omega^2x^2 &= E \end{aligned} \right\}$$

Values of  $E$  for which the boundary conditions of this second equation are satisfied are known as eigenvalues (or eigenenergies in this context), and the nontrivial solutions  $\psi(x)$  that satisfy this second equation are known as eigenfunctions (or eigenstates in this context). This ODE in  $x$  is known as the time-independent Schrödinger equation (TISE). Multiply both sides of it by  $\psi(x)$ .

$$-\frac{\hbar^2}{2m}\psi'' + \frac{1}{2}m\omega^2x^2\psi = E\psi$$

Since  $-\infty < x < \infty$ , the method of operator factorization will be used to solve the TISE.

$$\begin{aligned}
 \frac{1}{2m} \left( -\hbar^2 \frac{d^2 \psi}{dx^2} + m^2 \omega^2 x^2 \psi \right) &= E \psi \\
 \frac{1}{2m} \left[ \left( -i\hbar \frac{d}{dx} \right)^2 \psi + (m^2 \omega^2 x^2) \psi \right] &= E \psi \\
 \frac{1}{2m} (\hat{p}^2 + m^2 \omega^2 \hat{x}^2) \psi &= E \psi
 \end{aligned} \tag{1}$$

Because the position and momentum operators don't commute,

$$\begin{aligned}
 [\hat{x}, \hat{p}]f(x) &= (\hat{x}\hat{p} - \hat{p}\hat{x})f(x) \\
 &= \hat{x}\hat{p}f(x) - \hat{p}\hat{x}f(x) \\
 &= x \left( -i\hbar \frac{d}{dx} \right) f(x) - \left( -i\hbar \frac{d}{dx} \right) x f(x) \\
 &= -i\hbar x \frac{df}{dx} + i\hbar \frac{d}{dx} [x f(x)] \\
 &= \cancel{-i\hbar x \frac{df}{dx}} + i\hbar f(x) + i\hbar \cancel{x \frac{df}{dx}} \\
 &= i\hbar f(x) \quad \Rightarrow \quad [\hat{x}, \hat{p}] = i\hbar
 \end{aligned}$$

there are two equivalent ways of writing the Hamiltonian operator in equation (1).

$$\begin{aligned}
 \frac{1}{2m} (\hat{p}^2 + im\omega[\hat{x}, \hat{p}] + m^2 \omega^2 \hat{x}^2 - im\omega[\hat{x}, \hat{p}]) \psi &= E \psi \\
 \frac{1}{2m} [\hat{p}^2 + im\omega(\hat{x}\hat{p} - \hat{p}\hat{x}) + m^2 \omega^2 \hat{x}^2 - im\omega(i\hbar)] \psi &= E \psi \\
 \frac{1}{2m} (\hat{p}^2 + im\omega\hat{x}\hat{p} - im\omega\hat{p}\hat{x} + m^2 \omega^2 \hat{x}^2 + \hbar m\omega) \psi &= E \psi \\
 \frac{1}{2m} [(-i\hat{p} + m\omega\hat{x})(i\hat{p} + m\omega\hat{x}) + \hbar m\omega] \psi &= E \psi \\
 \hbar\omega \left( \frac{-i\hat{p} + m\omega\hat{x}}{\sqrt{2m\hbar\omega}} \frac{i\hat{p} + m\omega\hat{x}}{\sqrt{2m\hbar\omega}} + \frac{1}{2} \right) \psi &= E \psi
 \end{aligned}$$

$$\begin{aligned}
 \frac{1}{2m} (\hat{p}^2 - im\omega[\hat{x}, \hat{p}] + m^2 \omega^2 \hat{x}^2 + im\omega[\hat{x}, \hat{p}]) \psi &= E \psi \\
 \frac{1}{2m} [\hat{p}^2 - im\omega(\hat{x}\hat{p} - \hat{p}\hat{x}) + m^2 \omega^2 \hat{x}^2 + im\omega(i\hbar)] \psi &= E \psi \\
 \frac{1}{2m} (\hat{p}^2 - im\omega\hat{x}\hat{p} + im\omega\hat{p}\hat{x} + m^2 \omega^2 \hat{x}^2 - \hbar m\omega) \psi &= E \psi \\
 \frac{1}{2m} [(i\hat{p} + m\omega\hat{x})(-i\hat{p} + m\omega\hat{x}) - \hbar m\omega] \psi &= E \psi \\
 \hbar\omega \left( \frac{i\hat{p} + m\omega\hat{x}}{\sqrt{2m\hbar\omega}} \frac{-i\hat{p} + m\omega\hat{x}}{\sqrt{2m\hbar\omega}} - \frac{1}{2} \right) \psi &= E \psi
 \end{aligned}$$

Introducing the operators,

$$\hat{a}_+ = \frac{1}{\sqrt{2m\hbar\omega}}(-i\hat{p} + m\omega\hat{x})$$

$$\hat{a}_- = \frac{1}{\sqrt{2m\hbar\omega}}(i\hat{p} + m\omega\hat{x}),$$

the TISE can be written as

$$\hbar\omega \left( \hat{a}_+ \hat{a}_- + \frac{1}{2} \right) \psi = E\psi \quad \text{or} \quad \hbar\omega \left( \hat{a}_- \hat{a}_+ - \frac{1}{2} \right) \psi = E\psi.$$

According to Problem 2.3,  $E$  must be positive for there to be legitimate eigenstates. Expanding the left side of this first equation,

$$\hbar\omega \hat{a}_+ (\hat{a}_- \psi) + \frac{\hbar\omega}{2} \psi = E\psi,$$

we see that the smallest positive energy is  $E_0 = \hbar\omega/2$  and that the eigenstate associated with it (the ground state) satisfies

$$\hat{a}_- \psi_0 = 0.$$

Solve the resulting ODE for  $\psi_0$ .

$$\frac{1}{\sqrt{2m\hbar\omega}}(i\hat{p} + m\omega\hat{x})\psi_0 = 0$$

$$\frac{1}{\sqrt{2m\hbar\omega}} \left[ i \left( -i\hbar \frac{d}{dx} \right) + m\omega x \right] \psi_0 = 0$$

$$\frac{1}{\sqrt{2m\hbar\omega}} \left( \hbar \frac{d}{dx} + m\omega x \right) \psi_0 = 0$$

$$\hbar \frac{d\psi_0}{dx} + m\omega x \psi_0 = 0$$

$$\frac{\frac{d\psi_0}{dx}}{\psi_0} = -\frac{m\omega}{\hbar} x$$

$$\frac{d}{dx} \ln \psi_0 = -\frac{m\omega}{\hbar} x$$

$$\ln \psi_0 = -\frac{m\omega}{2\hbar} x^2 + C_0$$

$$\psi_0(x) = A_0 \exp \left( -\frac{m\omega}{2\hbar} x^2 \right)$$

$A_0$  is a normalization constant, which is chosen so that the integral of  $|\psi_0(x)|^2$  over the whole line is 1.

$$1 = \int_{-\infty}^{\infty} |\psi_0(x)|^2 dx$$

$$= \int_{-\infty}^{\infty} A_0^2 \exp \left( -\frac{m\omega}{\hbar} x^2 \right) dx$$

$$= 2A_0^2 \int_0^{\infty} \exp \left[ -\frac{x^2}{\left( \sqrt{\frac{\hbar}{m\omega}} \right)^2} \right] dx = 2A_0^2 \sqrt{\pi} \left( \frac{\sqrt{\frac{\hbar}{m\omega}}}{2} \right)$$

Solve for  $A_0$ .

$$A_0^2 \sqrt{\frac{\pi \hbar}{m\omega}} = 1$$

$$A_0 = \left(\frac{m\omega}{\pi \hbar}\right)^{1/4}$$

Therefore, the ground state of the harmonic oscillator is

$$\psi_0(x) = \left(\frac{m\omega}{\pi \hbar}\right)^{1/4} \exp\left(-\frac{m\omega}{2\hbar}x^2\right).$$

Since the left sides of the TISE are equivalent,

$$\hat{a}_+ \hat{a}_- + \frac{1}{2} = \hat{a}_- \hat{a}_+ - \frac{1}{2}$$

$$\hat{a}_- \hat{a}_+ - \hat{a}_+ \hat{a}_- = 1$$

$$[\hat{a}_-, \hat{a}_+] = 1.$$

$\hat{a}_+$  and  $\hat{a}_-$  are known as the promotion and demotion operators, respectively, because they raise or lower a given eigenstate by 1 level. The energy of this new state can be calculated.

$$\begin{aligned} \hat{H}(\hat{a}_+ \psi) &= \hbar\omega \left(\hat{a}_+ \hat{a}_- + \frac{1}{2}\right) \hat{a}_+ \psi & \hat{H}(\hat{a}_- \psi) &= \hbar\omega \left(\hat{a}_- \hat{a}_+ - \frac{1}{2}\right) \hat{a}_- \psi \\ &= \hbar\omega \left(\hat{a}_+ \hat{a}_- \hat{a}_+ + \frac{1}{2} \hat{a}_+\right) \psi & &= \hbar\omega \left(\hat{a}_- \hat{a}_+ \hat{a}_- - \frac{1}{2} \hat{a}_-\right) \psi \\ &= \hbar\omega \hat{a}_+ \left(\hat{a}_- \hat{a}_+ + \frac{1}{2}\right) \psi & &= \hbar\omega \hat{a}_- \left(\hat{a}_+ \hat{a}_- - \frac{1}{2}\right) \psi \\ &= \hbar\omega \hat{a}_+ \left(\hat{a}_+ \hat{a}_- + 1 + \frac{1}{2}\right) \psi & &= \hbar\omega \hat{a}_- \left(\hat{a}_- \hat{a}_+ - 1 - \frac{1}{2}\right) \psi \\ &= \hat{a}_+ \left[\hbar\omega \left(\hat{a}_+ \hat{a}_- + \frac{1}{2}\right) + \hbar\omega\right] \psi & &= \hat{a}_- \left[\hbar\omega \left(\hat{a}_- \hat{a}_+ - \frac{1}{2}\right) - \hbar\omega\right] \psi \\ &= \hat{a}_+ \left[\hbar\omega \left(\hat{a}_+ \hat{a}_- + \frac{1}{2}\right) \psi + \hbar\omega \psi\right] & &= \hat{a}_- \left[\hbar\omega \left(\hat{a}_- \hat{a}_+ - \frac{1}{2}\right) \psi - \hbar\omega \psi\right] \\ &= \hat{a}_+ (E\psi + \hbar\omega \psi) & &= \hat{a}_- (E\psi - \hbar\omega \psi) \\ &= \hat{a}_+ (E + \hbar\omega) \psi & &= \hat{a}_- (E - \hbar\omega) \psi \\ &= (E + \hbar\omega) \hat{a}_+ \psi & &= (E - \hbar\omega) \hat{a}_- \psi \end{aligned}$$

The energy of the first excited state is then

$$E_1 = \frac{\hbar\omega}{2} + \hbar\omega = \frac{3}{2}\hbar\omega,$$

and the eigenstate associated with it is

$$\begin{aligned}
 \psi_1(x) &= A_1(\hat{a}_+\psi_0) \\
 &= \frac{A_1}{\sqrt{2m\hbar\omega}}(-i\hat{p} + m\omega\hat{x})\psi_0(x) \\
 &= \frac{A_1}{\sqrt{2m\hbar\omega}}\left[-i\left(-i\hbar\frac{d}{dx}\right) + m\omega x\right]\psi_0(x) \\
 &= \frac{A_1}{\sqrt{2m\hbar\omega}}\left(-\hbar\frac{d\psi_0}{dx} + m\omega x\psi_0\right) \\
 &= \frac{A_1}{\sqrt{2m\hbar\omega}}\left\{-\hbar\frac{d}{dx}\left[\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{m\omega}{2\hbar}x^2\right)\right] + m\omega x\left[\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{m\omega}{2\hbar}x^2\right)\right]\right\} \\
 &= \frac{A_1}{\sqrt{2m\hbar\omega}}\left[-\hbar\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{m\omega}{2\hbar}x^2\right)\left(-\frac{m\omega}{\hbar}x\right) + m\omega x\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{m\omega}{2\hbar}x^2\right)\right] \\
 &= \frac{A_1}{\sqrt{2m\hbar\omega}}\left[-\hbar\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left(-\frac{m\omega}{\hbar}\right) + m\omega\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\right]x\exp\left(-\frac{m\omega}{2\hbar}x^2\right) \\
 &= \frac{A_1}{\sqrt{2m\hbar\omega}}\left[2m\omega\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\right]x\exp\left(-\frac{m\omega}{2\hbar}x^2\right) \\
 &= A_1\sqrt{\frac{2m\omega}{\hbar}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}x\exp\left(-\frac{m\omega}{2\hbar}x^2\right) \\
 &= A_1\left(\frac{4m^3\omega^3}{\pi\hbar^3}\right)^{1/4}x\exp\left(-\frac{m\omega}{2\hbar}x^2\right).
 \end{aligned}$$

$A_1$  is a normalization constant, which is chosen so that the integral of  $|\psi_1(x)|^2$  over the whole line is 1.

$$\begin{aligned}
 1 &= \int_{-\infty}^{\infty} |\psi_1(x)|^2 dx \\
 &= \int_{-\infty}^{\infty} A_1^2 \left(\frac{4m^3\omega^3}{\pi\hbar^3}\right)^{1/2} x^2 \exp\left(-\frac{m\omega}{\hbar}x^2\right) dx \\
 &= 2A_1^2 \left(\frac{4m^3\omega^3}{\pi\hbar^3}\right)^{1/2} \int_0^{\infty} x^2 \exp\left[-\frac{x^2}{\left(\sqrt{\frac{\hbar}{m\omega}}\right)^2}\right] dx \\
 &= 2A_1^2 \left(\frac{4m^3\omega^3}{\pi\hbar^3}\right)^{1/2} \cdot \sqrt{\pi} \frac{2!}{1!} \left(\frac{\sqrt{\frac{\hbar}{m\omega}}}{2}\right)^3 \\
 &= A_1^2
 \end{aligned}$$

Solve for  $A_1$ .

$$A_1 = 1$$

Therefore, the first excited state of the harmonic oscillator is

$$\psi_1(x) = \left(\frac{4m^3\omega^3}{\pi\hbar^3}\right)^{1/4} x \exp\left(-\frac{m\omega}{2\hbar}x^2\right).$$

The energy of the second excited state is

$$E_2 = \frac{3}{2}\hbar\omega + \hbar\omega = \frac{5}{2}\hbar\omega,$$

and the eigenstate associated with it is

$$\begin{aligned}\psi_2(x) &= A_2(\hat{a}_+\psi_1) \\ &= \frac{A_2}{\sqrt{2m\hbar\omega}}(-i\hat{p} + m\omega\hat{x})\psi_1(x) \\ &= \frac{A_2}{\sqrt{2m\hbar\omega}}\left[-i\left(-i\hbar\frac{d}{dx}\right) + m\omega x\right]\psi_1(x) \\ &= \frac{A_2}{\sqrt{2m\hbar\omega}}\left(-\hbar\frac{d\psi_1}{dx} + m\omega x\psi_1\right) \\ &= \frac{A_2}{\sqrt{2m\hbar\omega}}\left\{-\hbar\frac{d}{dx}\left[\left(\frac{4m^3\omega^3}{\pi\hbar^3}\right)^{1/4}x\exp\left(-\frac{m\omega}{2\hbar}x^2\right)\right] + m\omega x\left[\left(\frac{4m^3\omega^3}{\pi\hbar^3}\right)^{1/4}x\exp\left(-\frac{m\omega}{2\hbar}x^2\right)\right]\right\} \\ &= \frac{A_2}{\sqrt{2m\hbar\omega}}\left\{-\hbar\left(\frac{4m^3\omega^3}{\pi\hbar^3}\right)^{1/4}\left[\exp\left(-\frac{m\omega}{2\hbar}x^2\right) + x\exp\left(-\frac{m\omega}{2\hbar}x^2\right)\left(-\frac{m\omega}{\hbar}x\right)\right] \right. \\ &\quad \left. + m\omega\left(\frac{4m^3\omega^3}{\pi\hbar^3}\right)^{1/4}x^2\exp\left(-\frac{m\omega}{2\hbar}x^2\right)\right\} \\ &= \frac{A_2}{\sqrt{2m\hbar\omega}}\left(\frac{4m^3\omega^3}{\pi\hbar^3}\right)^{1/4}\left[-\hbar\exp\left(-\frac{m\omega}{2\hbar}x^2\right) + m\omega x^2\exp\left(-\frac{m\omega}{2\hbar}x^2\right) + m\omega x^2\exp\left(-\frac{m\omega}{2\hbar}x^2\right)\right] \\ &= \frac{A_2}{\sqrt{2m\hbar\omega}}\left(\frac{4m^3\omega^3}{\pi\hbar^3}\right)^{1/4}(2m\omega x^2 - \hbar)\exp\left(-\frac{m\omega}{2\hbar}x^2\right) \\ &= A_2\sqrt{\frac{\hbar}{2m\omega}}\left(\frac{4m^3\omega^3}{\pi\hbar^3}\right)^{1/4}\left(\frac{2m\omega}{\hbar}x^2 - 1\right)\exp\left(-\frac{m\omega}{2\hbar}x^2\right) \\ &= A_2\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left(\frac{2m\omega}{\hbar}x^2 - 1\right)\exp\left(-\frac{m\omega}{2\hbar}x^2\right).\end{aligned}$$

$A_2$  is a normalization constant, which is chosen so that the integral of  $|\psi_2(x)|^2$  over the whole line is 1.

$$\begin{aligned}1 &= \int_{-\infty}^{\infty} |\psi_2(x)|^2 dx \\ &= \int_{-\infty}^{\infty} A_2^2 \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \left(\frac{2m\omega}{\hbar}x^2 - 1\right)^2 \exp\left(-\frac{m\omega}{\hbar}x^2\right) dx \\ &= 2A_2^2 \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \int_0^{\infty} \left(\frac{2m\omega}{\hbar}x^2 - 1\right)^2 \exp\left(-\frac{m\omega}{\hbar}x^2\right) dx\end{aligned}$$

Make the following substitution.

$$\begin{aligned}\xi &= \sqrt{\frac{m\omega}{\hbar}}x \\ d\xi &= \sqrt{\frac{m\omega}{\hbar}}dx \quad \rightarrow \quad dx = \sqrt{\frac{\hbar}{m\omega}}d\xi\end{aligned}$$

Consequently,

$$\begin{aligned}
 1 &= 2A_2^2 \left( \frac{m\omega}{\pi\hbar} \right)^{1/2} \int_0^\infty (2\xi^2 - 1)^2 e^{-\xi^2} \left( \sqrt{\frac{\hbar}{m\omega}} d\xi \right) \\
 &= \frac{2A_2^2}{\pi^{1/2}} \int_0^\infty (2\xi^2 - 1)^2 e^{-\xi^2} d\xi \\
 &= \frac{2A_2^2}{\pi^{1/2}} \int_0^\infty (4\xi^4 - 4\xi^2 + 1) e^{-\xi^2} d\xi \\
 &= \frac{2A_2^2}{\pi^{1/2}} \left( 4 \int_0^\infty \xi^4 e^{-\xi^2} d\xi - 4 \int_0^\infty \xi^2 e^{-\xi^2} d\xi + \int_0^\infty e^{-\xi^2} d\xi \right) \\
 &= \frac{2A_2^2}{\pi^{1/2}} \left[ 4 \cdot \sqrt{\pi} \frac{4!}{2!} \left( \frac{1}{2} \right)^5 - 4 \cdot \sqrt{\pi} \frac{2!}{1!} \left( \frac{1}{2} \right)^3 + \sqrt{\pi} \left( \frac{1}{2} \right) \right] \\
 &= 2A_2^2.
 \end{aligned}$$

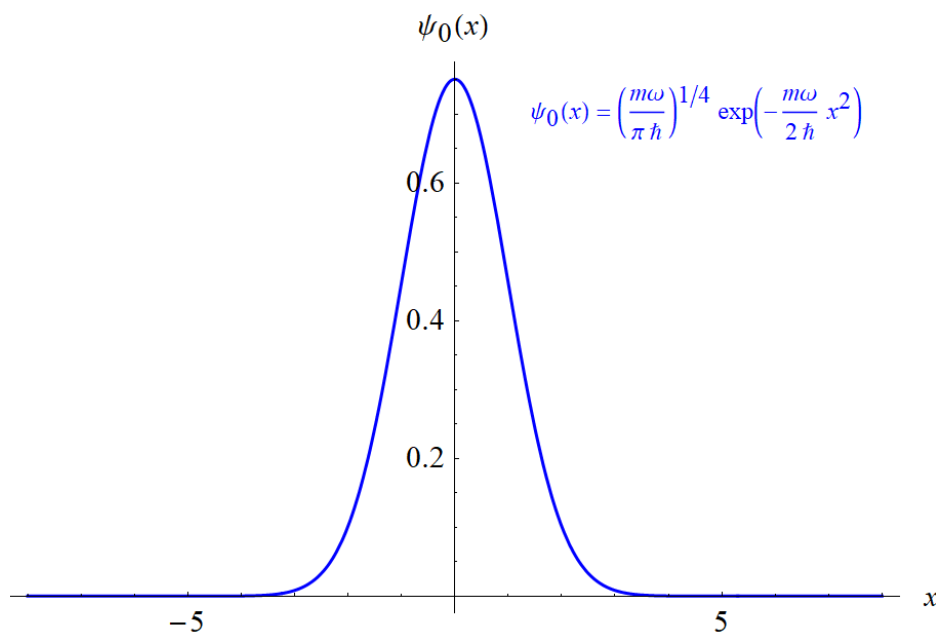
Solve for  $A_2$ .

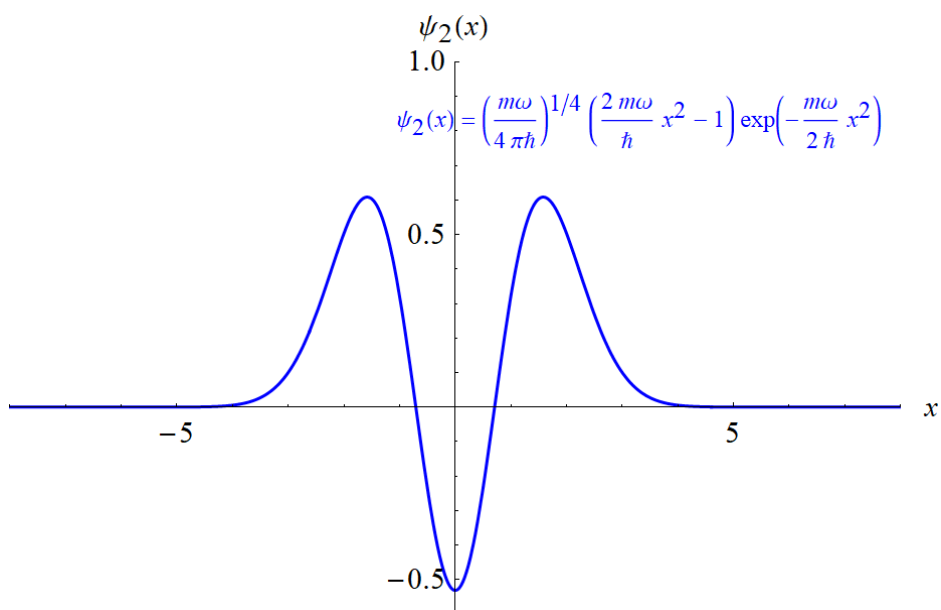
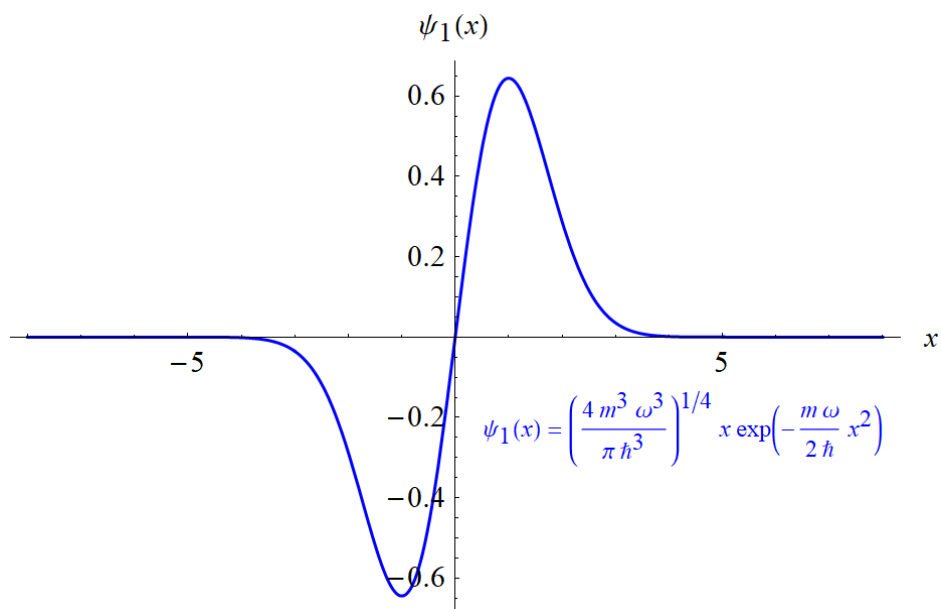
$$A_2 = \frac{1}{\sqrt{2}}$$

Therefore, the second excited state of the harmonic oscillator is

$$\psi_2(x) = \left( \frac{m\omega}{4\pi\hbar} \right)^{1/4} \left( \frac{2m\omega}{\hbar} x^2 - 1 \right) \exp \left( -\frac{m\omega}{2\hbar} x^2 \right).$$

We could keep using the promotion operator to get  $\psi_3(x)$ ,  $\psi_4(x)$ , and so on if we wanted to. Below are plots of the first three eigenstates versus  $x$  for the special case that  $m\omega/\hbar = 1$ .





$\psi_0(x)$  and  $\psi_1(x)$  are orthogonal because the integral of an odd function over a symmetric interval is zero.

$$\begin{aligned} \int_{-\infty}^{\infty} \psi_0^*(x) \psi_1(x) dx &= \int_{-\infty}^{\infty} \left( \frac{m \omega}{\pi \hbar} \right)^{1/4} \exp\left(-\frac{m \omega}{2 \hbar} x^2\right) \left( \frac{4 m^3 \omega^3}{\pi \hbar^3} \right)^{1/4} x \exp\left(-\frac{m \omega}{2 \hbar} x^2\right) dx \\ &= \sqrt{\frac{2}{\pi}} \frac{m \omega}{\hbar} \int_{-\infty}^{\infty} x \exp\left(-\frac{m \omega}{\hbar} x^2\right) dx \\ &= 0 \end{aligned}$$

$\psi_1(x)$  and  $\psi_2(x)$  are orthogonal because the integral of an odd function over a symmetric interval is zero.

$$\begin{aligned}\int_{-\infty}^{\infty} \psi_1^*(x) \psi_2(x) dx &= \int_{-\infty}^{\infty} \left( \frac{4m^3 \omega^3}{\pi \hbar^3} \right)^{1/4} x \exp\left(-\frac{m\omega}{2\hbar} x^2\right) \left( \frac{m\omega}{4\pi \hbar} \right)^{1/4} \left( \frac{2m\omega}{\hbar} x^2 - 1 \right) \exp\left(-\frac{m\omega}{2\hbar} x^2\right) dx \\ &= \frac{1}{\sqrt{\pi}} \frac{m\omega}{\hbar} \int_{-\infty}^{\infty} x \left( \frac{2m\omega}{\hbar} x^2 - 1 \right) \exp\left(-\frac{m\omega}{\hbar} x^2\right) dx \\ &= 0\end{aligned}$$

Now check the orthogonality of  $\psi_0(x)$  and  $\psi_2(x)$ .

$$\begin{aligned}\int_{-\infty}^{\infty} \psi_0^*(x) \psi_2(x) dx &= \int_{-\infty}^{\infty} \left( \frac{m\omega}{\pi \hbar} \right)^{1/4} \exp\left(-\frac{m\omega}{2\hbar} x^2\right) \left( \frac{m\omega}{4\pi \hbar} \right)^{1/4} \left( \frac{2m\omega}{\hbar} x^2 - 1 \right) \exp\left(-\frac{m\omega}{2\hbar} x^2\right) dx \\ &= \sqrt{\frac{m\omega}{2\pi \hbar}} \int_{-\infty}^{\infty} \left( \frac{2m\omega}{\hbar} x^2 - 1 \right) \exp\left(-\frac{m\omega}{\hbar} x^2\right) dx \\ &= 2\sqrt{\frac{m\omega}{2\pi \hbar}} \int_0^{\infty} \left( \frac{2m\omega}{\hbar} x^2 - 1 \right) \exp\left(-\frac{m\omega}{\hbar} x^2\right) dx\end{aligned}$$

Make the following substitution.

$$\begin{aligned}\xi &= \sqrt{\frac{m\omega}{\hbar}} x \\ d\xi &= \sqrt{\frac{m\omega}{\hbar}} dx \quad \rightarrow \quad dx = \sqrt{\frac{\hbar}{m\omega}} d\xi\end{aligned}$$

Consequently,

$$\begin{aligned}\int_{-\infty}^{\infty} \psi_0^*(x) \psi_2(x) dx &= 2\sqrt{\frac{m\omega}{2\pi \hbar}} \int_0^{\infty} (2\xi^2 - 1) e^{-\xi^2} \left( \sqrt{\frac{\hbar}{m\omega}} d\xi \right) \\ &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} (2\xi^2 - 1) e^{-\xi^2} d\xi \\ &= \sqrt{\frac{2}{\pi}} \left( 2 \int_0^{\infty} \xi^2 e^{-\xi^2} d\xi - \int_0^{\infty} e^{-\xi^2} d\xi \right) \\ &= \sqrt{\frac{2}{\pi}} \left[ 2 \cdot \sqrt{\pi} \frac{2!}{1!} \left( \frac{1}{2} \right)^3 - \sqrt{\pi} \left( \frac{1}{2} \right) \right] \\ &= \sqrt{\frac{2}{\pi}} \left( \frac{\sqrt{\pi}}{2} - \frac{\sqrt{\pi}}{2} \right) \\ &= 0,\end{aligned}$$

which means  $\psi_0(x)$  and  $\psi_2(x)$  are orthogonal as well. With the eigenenergies of the first three eigenstates known, the ODE in  $t$  can be solved to determine the corresponding  $\phi(t)$  in each case.

$$\begin{array}{lll} i\hbar \frac{\phi_0'(t)}{\phi_0(t)} = \frac{\hbar\omega}{2} & i\hbar \frac{\phi_1'(t)}{\phi_1(t)} = \frac{3}{2}\hbar\omega & i\hbar \frac{\phi_2'(t)}{\phi_2(t)} = \frac{5}{2}\hbar\omega \\ \phi_0(t) = D_0 e^{-i\omega t/2} & \phi_1(t) = D_1 e^{-3i\omega t/2} & \phi_2(t) = D_2 e^{-5i\omega t/2}\end{array}$$

According to the principle of superposition, the general solution to Schrödinger's equation is a linear combination of the product solutions (or stationary states in this context)

$\Psi_n(x, t) = \psi_n(x)\phi_n(t)$  over all the eigenvalues.

$$\begin{aligned}
 \Psi(x, t) &= \sum_{n=0}^{\infty} B_n \Psi_n(x, t) \\
 &= B_0 \Psi_0(x, t) + B_1 \Psi_1(x, t) + B_2 \Psi_2(x, t) + \cdots \\
 &= B_0 \psi_0(x) \phi_0(t) + B_1 \psi_1(x) \phi_1(t) + B_2 \psi_2(x) \phi_2(t) + \cdots \\
 &= B_0 \left( \frac{m\omega}{\pi\hbar} \right)^{1/4} \exp \left( -\frac{m\omega}{2\hbar} x^2 \right) e^{-i\omega t/2} \\
 &\quad + B_1 \left( \frac{4m^3\omega^3}{\pi\hbar^3} \right)^{1/4} x \exp \left( -\frac{m\omega}{2\hbar} x^2 \right) e^{-3i\omega t/2} \\
 &\quad + B_2 \left( \frac{m\omega}{4\pi\hbar} \right)^{1/4} \left( \frac{2m\omega}{\hbar} x^2 - 1 \right) \exp \left( -\frac{m\omega}{2\hbar} x^2 \right) e^{-5i\omega t/2} + \cdots
 \end{aligned}$$

With a provided initial condition  $\Psi(x, 0)$  for the wave function, these coefficients  $B_n$  could be determined.